Relativistic Brownian motion: From a microscopic binary collision model to the Langevin equation

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The Langevin equation (LE) for the one-dimensional relativistic Brownian motion is derived from a microscopic collision model. The model assumes that a heavy pointlike Brownian particle interacts with the lighter heat bath particles via elastic hard-core collisions. First, the commonly known, nonrelativistic LE is deduced from this model, by taking into account the nonrelativistic conservation laws for momentum and kinetic energy. Subsequently, this procedure is generalized to the relativistic case. There, it is found that the relativistic stochastic force is still δ correlated (white noise) but no longer corresponds to a Gaussian white noise process. Explicit results for the friction and momentum-space diffusion coefficients are presented and discussed.

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I. INTRODUCTION

The theories of nonrelativistic Brownian motion and special relativity were introduced more than 100 years ago [1–7]. Since then, they have become cornerstones for our understanding of a wide range of physical processes [8–12]. This fact notwithstanding, the unification of both concepts still poses a theoretical challenge nowadays (classical references are [13–18]; recent contributions include [19–32]; potential applications in high-energy physics and astrophysics are considered in [33–38]). The relatively slow progress in this field can be attributed to the severe difficulties that arise when one tries to describe N-body systems in a relativistically consistent manner [39–43]. For this reason, the derivation of relativistic Langevin equations (LEs) from an underlying microscopic model has remained an unsolved issue until now.[1] However, in the present paper we aim to provide a solution to this problem.

More precisely, by considering quasielastic, binary collisions between the Brownian and heat bath particles [48,49] we are able to treat the heat bath in a fully relativistic manner without having to account for the exact details of the relativistic N-body interactions. As shown in Sec. II, for a nonrelativistic framework this approach yields the well-known nonrelativistic LE with Gaussian white noise as well as the correct fluctuation-dissipation theorem (the Einstein-Sutherland relation [1]). In Sec. III, the method is transferred to the relativistic case, leading to the main result of this paper, the relativistic LE [32]. Remarkably, the relativistic stochastic force is also δ correlated ("white") but no longer of Gaussian (or Wiener [50]) type. Compared with the nonrelativistic Brownian motion, this is the most important difference. Furthermore, we obtain explicit representations of friction and (momentum) diffusion coefficients in terms of expectation values with respect to the heat bath distribution (see also the Appendix).

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[1]For the nonrelativistic Brownian motion, this problem was solved by Bogolyubov [44], Magalinskii [45], Ford *et al.* [46], and Zwanzig [47], who considered a bath of harmonic oscillators.

II. NONRELATIVISTIC BROWNIAN MOTIONS

The objective of this section is to recover the well-known nonrelativistic LEs from a simple microscopic collision model for Brownian motions. As shown by several authors in the past [44–47,51], nonrelativistic LEs can also be derived by considering a bath of harmonic oscillators with canonical phase space distribution. Unfortunately, it is problematic to transfer this approach to the relativistic case, because any instantaneous linear (or nonlinear) interactions between Brownian and heat particles would violate the basic principles of special relativity [41,42]. To circumvent this problem, we will pursue a different method here, using only the (non)relativistic microscopic conservation laws for energy and momentum, respectively, known to hold for elastic pointlike, binary collisions (contact interactions [42]). Conceptually, our approach is related to that of Pechukas [48] and Pechukas and Tsonchev [49], who considered a similar model in the context of nonrelativistic quantum Brownian motion [52]. Analogous approaches are also known from unimolecular rate theory; see, e.g., Sec. V in [53].

A. Microscopic model

For the sake of simplicity only, we will restrict ourselves throughout to the one-dimensional (1D) case. Generalizations to higher space dimensions are in principle straightforward, but certain calculations will become much more cumbersome (see the comments at the end of the Appendix). To start out, consider the following situation in the inertial laboratory frame Σ_0: A large one-dimensional box volume $\mathcal{V}\equiv[-L/2,L/2]$ contains an ideal nonrelativistic gas, consisting of N small pointlike particles with identical masses m. The gas particles—referred to as the heat bath hereafter—surround a Brownian particle of mass $M \gg m$. Because of frequent elastic collisions with heat bath particles, the Brownian particle performs stochastic motions.

1. Heat bath

The coordinates and momenta of the heat bath particles are denoted by $x_r \in [-L/2,L/2]$ and $p_r \in (-\infty;\infty)$, respectively, where $r=1,\ldots,N$. As usual, we make the following simplifying assumption concerning the heat bath. The probability density function (PDF) of the heat bath particles is a spatially homogeneous Maxwell distribution, i.e., at each time $t>0$, the PDF reads

$$f_{\mathrm{b}}^{N}(x_1, \ldots, p_N) = \left(\frac{\lambda}{L}\right)^N \prod_{r=1}^{N} \exp\left(-\frac{p_r^2}{2mkT}\right), \tag{1}$$

where k is the Boltzmann constant, T the temperature, and $\lambda=(2\pi mkT)^{-1/2}$. Thus, it is implicitly assumed that the heat bath particles are independently and identically distributed; and the collisions with the Brownian particle do not significantly alter the bath distribution. These assumptions are justified, if the collisions between the gas particles rapidly reestablish a spatially homogeneous bath distribution.

2. Kinematics of single-collision events

The momentum and energy balance per (elastic) collision reads

$$E + \epsilon = \hat{E} + \hat{\epsilon}, \quad P + p = \hat{P} + \hat{p}. \tag{2}$$

Here and below, capital letters refer to the Brownian particle and lower-case letters to particles forming the heat bath; quantities without (with) carets refer to the state before (after) the collision. In the nonrelativistic case, we have, e.g., before the collision

$$P = MV, \quad p = mv, \quad E = \frac{P^2}{2M}, \quad \epsilon = \frac{p^2}{2m}, \tag{3}$$

where v and V denote the velocities. Taking into account conservation of both momentum and (kinetic) energy, one finds that the change $\Delta P \equiv \hat{P} - P$ of the Brownian particle's momentum per single collision is given by

$$\Delta P = \frac{-2m}{M+m}P + \frac{2M}{M+m}p. \tag{4}$$

B. Derivation of the Langevin equation

The total momentum change δP of the Brownian particle within the time interval τ can be written as

$$\delta P(t) \equiv P(t+\tau) - P(t) = \sum_{r=1}^{N} \Delta P_r I_r(t,\tau), \tag{5}$$

where $I_r(t,\tau) \in \{0,1\}$ is the indicator function for a collision with the heat bath particle r during the interval $[t, t+\tau]$; i.e., $I_r(t,\tau)=1$ if a collision has occurred, and, otherwise, $I_r(t,\tau)=0$. In the 1D case, the collision indicator can be written explicitly as

$$\begin{aligned} I_r(t,\tau) = {} & \Theta(X - x_r)\Theta(x_r' - X')\Theta(v_r - V) \\ & + \Theta(x_r - X)\Theta(X' - x_r')\Theta(V - v_r), \end{aligned} \tag{6a}$$

where $X=X(t)$, $x_r=x_r(t)$, and

$$X' = X + V\tau, \quad x_r' = x_r + v_r\tau \tag{6b}$$

are the projected particle positions at time $t+\tau$. The Heaviside function is defined by

$$\Theta(x) = \begin{cases} 0, & x<0, \\ 1/2, & x=0, \\ 1, & x>0. \end{cases}$$

The expectation of the collision indicator with respect to the bath distribution, denoted by $\langle I_r(t,\tau)\rangle_{\mathrm{b}}$, gives the probability that the bath particle r collides with the Brownian particle between t and $t+\tau$. As shown in the Appendix, in the limit $\tau\to 0$, one finds

$$\langle I_r(t,\tau)\rangle_{\mathrm{b}} = \tilde{C}(V)\frac{\tau}{L} = C(P)\frac{\tau}{L}, \tag{7a}$$

with the function $C(P)=\tilde{C}(V(P))$ given by the integral formula

$$\tilde{C}(V) \equiv \frac{1}{2}\int_V^{\infty} dv_r (v_r - V)\tilde{f}_{\mathrm{b}}^{1}(v_r) + \frac{1}{2}\int_{-\infty}^{V} dv_r (V - v_r)\tilde{f}_{\mathrm{b}}^{1}(v_r). \tag{7b}$$

Here, $\tilde{f}_{\mathrm{b}}^{1}(v_r)$ is the one-particle velocity PDF of a heat bath particle. We anticipate that Eqs.(6) and (7) remain valid in the relativistic case as well, but then one has to insert the relativistic bath distribution in Eq. (7b).

However, in order to recover from Eqs. (5)–(7) the well-known nonrelativistic LE, we still have to make a number of simplifying assumptions.

(i) We assume that the time interval τ is sufficiently small, so that $|\delta P/P| \ll 1$. In particular, τ is supposed to be so small that there occurs at most only one collision between the Brownian particle and a specific heat bath particle r. On the other hand, the time interval τ should still be large enough that the total number of collisions within τ is larger than 1. These requirements can be satisfied simultaneously only if $m/M \ll 1$.

(ii) We assume that collisions occurring within $[t, t+\tau]$ can be viewed as independent events.

(iii) Finally, we will (have to) assume that

$$\langle [p_r I_r(t,\tau)]^j\rangle_{\mathrm{b}} = \langle p_r^j I_r(t,\tau)\rangle_{\mathrm{b}} \simeq \langle p_r^j\rangle_{\mathrm{b}}\langle I_r(t,\tau)\rangle_{\mathrm{b}} = \langle p_r^j\rangle_{\mathrm{b}} C(P)\frac{\tau}{L} \tag{8}$$

for $j=1,2,\ldots$. Given the explicit representation of the indicator function (6a), it is in principle straightforward to check the quality of the approximation (8), if a bath distribution has been specified.

As we shall see immediately, the assumptions (i)–(iii) are necessary and sufficient for deriving the well-known nonrelativistic LE from Eqs. (5)–(7). Upon inserting Eq. (4) into (5) and dividing by τ we find

$$\frac{\delta P(t)}{\tau} \simeq -\left[\frac{1}{\tau}\sum_{r=1}^{N}\frac{2m}{m+M}I_r(t,\tau)\right]P + \frac{1}{\tau}\sum_{r=1}^{N}\frac{2M}{M+m}p_r I_r(t,\tau). \tag{9}$$

The first term on the right-hand side (RHS) in Eq. (9) can be identified as the "friction" term, whereas the second term represents "noise." On the RHS of Eq. (9), it was assumed that for each collision occurring within $[t,t+\tau]$, the "initial" momentum of the Brownian particle is approximately equal to some suitably chosen value $P(t')$ with $t' \in [t,t+\tau]$; cf. the assumption (i) above and the discussion at the end of this section.

The next step en route to the conventional LE consists in replacing the expression in square brackets in Eq. (9) by the *averaged* friction coefficient

$$\nu_0(P) \equiv \frac{1}{\tau}\sum_{r=1}^{N}\frac{2m}{m+M}\langle I_r(t,\tau)\rangle_{\mathrm{b}}. \tag{10a}$$

Since it was assumed that the heat bath particles are independently and identically distributed, we can rewrite this as

$$\nu_0(P) = \frac{N}{\tau}\frac{2m}{m+M}\langle I_r(t,\tau)\rangle_{\mathrm{b}}, \tag{10b}$$

for some $r \in \{1,\ldots,N\}$. The coefficient ν_0 can be interpreted as an average collision rate weighted by some mass ratio. Inserting Eq. (7a) into Eq. (10b) yields

$$\nu_0(P) = n_{\mathrm{b}}\frac{2m}{m+M}C(P), \tag{11a}$$

where $n_{\mathrm{b}}=N/L$ is the density of the bath particles. In the case of the Maxwell distribution, we can evaluate the integral (7b), and find

$$C(P) = \left(\frac{kT}{2\pi m}\right)^{1/2}\exp\left[-\frac{m}{2kT}\left(\frac{P}{M}\right)^2\right] + \frac{P}{2M}\operatorname{erf}\left[\left(\frac{m}{2kT}\right)^{1/2}\frac{P}{M}\right]. \tag{11b}$$

In particular, setting (see the Appendix)

$$C(P) \approx C(0) = \left(\frac{kT}{2\pi m}\right)^{1/2} \tag{12}$$

corresponds to the commonly used Stokes approximation.

It then remains to analyze the "noise force"

$$\xi(P,t) \equiv \frac{1}{\tau}\sum_{r=1}^{N}\frac{2M}{M+m}p_r I_r(t,\tau), \tag{13}$$

corresponding to the last term in Eq. (9). The momentum dependence of the noise enters through the implicit P dependence of the collision indicator functions $I_r(t,\tau)$. To keep subsequent formulas as compact as possible, we shall use the abbreviation $\xi(t)\equiv\xi(P,t)$ in the remainder. Then, averaging over the bath distribution f_{b}^N and using Eqs. (8), we find for the mean value

$$\langle\xi(t)\rangle_{\mathrm{b}} = 0. \tag{14a}$$

Furthermore, assuming mutual independence of the collisions, the correlation function is obtained as

$$\begin{aligned}\langle\xi(t)\xi(s)\rangle_{\mathrm{b}} &= \frac{\delta_{ts}}{\tau^2}\left(\frac{2M}{M+m}\right)^2\sum_{r=1}^{N}\langle p_r^2 I_r^2(t,\tau)\rangle_{\mathrm{b}} \\ &\overset{(8)}{\simeq} \frac{\delta_{ts}}{\tau^2}\left(\frac{2M}{M+m}\right)^2\sum_{r=1}^{N} mkT\langle I_r(t,\tau)\rangle_{\mathrm{b}} \\ &\overset{(10a)}{=} \frac{\delta_{ts}}{\tau}\left(\frac{2M^2}{M+m}\right)\nu_0 kT,\end{aligned} \tag{14b}$$

with $\delta_{ts}\in\{0,1\}$ denoting the Kronecker symbol. To obtain the second line, we have used that $I_r^2(t,\tau)=I_r(t,\tau)$, and the simplifying assumption (8) that $I_r(t,\tau)$ and p_r are (approximately) independent random variables with respect to the bath distribution.

Similar to Eq. (14b), also the higher correlation functions are determined by the corresponding moments of the Gaussian marginal bath distribution (1). Thus, under the above assumptions (i)–(iii), the nonrelativistic stochastic force $\xi(t)$ corresponds to Gaussian white noise (or a Wiener process [50], respectively).

Finally, by substituting ν_0 from Eqs. (11) for the expression in square brackets in Eq. (9) and formally letting $\tau\to 0$, we recover from Eq. (9) the well-known nonrelativistic LE [51,54,55]

$$\dot{P} = -\nu_0(P)P + \xi(t), \tag{15a}$$

where $\xi(t)\equiv\xi(P,t)$ is a momentum-dependent Gaussian white noise force, characterized by

$$\langle\xi(t)\rangle = 0, \tag{15b}$$

$$\langle\xi(t)\xi(s)\rangle = 2D_0(P)\delta(t-s), \tag{15c}$$

with (momentum-space) diffusion coefficient

$$D_0(P) = \frac{M^2}{M+m}\nu_0(P)kT. \tag{15d}$$

To obtain Eq. (15c), we used that $\delta_{st}/\tau\to\delta(t-s)$ for $\tau\to 0$, where $\delta(t-s)$ is the Dirac function.

In the limit $m/M\to 0$, Eq. (15d) reduces to the standard fluctuation-dissipation theorem $D_0=M\nu_0 kT$ [1,56]. However, ν_0 and D_0 are constants only if one adopts the Stokes approximation (12) (see the Appendix). If one goes beyond the Stokes approximation, then the noise in Eqs. (15) becomes multiplicative with respect to P, and, therefore, Eqs. (15) must be complemented by a discretization rule in this case [54,57–64]. As discussed in [54,62–64], only for the post-point-discretization rule, corresponding to the choice $\nu_0(P)=\nu_0[P(t+\tau)]$ and $D_0(P)=D_0[P(t+\tau)]$ on the RHS of Eq. (15a), one recovers the Maxwellian PDF

$$\Phi_\infty(P) = \left(\frac{1}{2\pi MkT}\right)^{1/2} \exp\left(-\frac{P^2}{2MkT}\right) \tag{16}$$

as the stationary momentum distribution of the Brownian particle in the limit $t \to \infty$ (assuming that $m/M \to 0$).

III. RELATIVISTIC BROWNIAN MOTIONS

We shall now apply an analogous reasoning to obtain a relativistic LE. For this purpose we consider an inertial (laboratory) frame Σ_0 with time coordinate t, as, e.g., measured by an atomic clock resting in Σ_0.

A. Microscopic model

The basic constituents of the microscopic model are the same as those outlined in Sec. II A, but in addition we now have to consider a relativistic heat bath distribution and must consistently take into account the relativistic collision kinematics.

1. Relativistic heat bath

In the relativistic case, we postulate, analogously to Eq. (1), that with respect to Σ_0 the heat bath distribution is stationary, spatially homogeneous, and independent, so that the PDF can be written in the product form

$$f_\mathrm{b}^N(x_1, \ldots, p_N) = L^{-N} \prod_{r=1}^N f_\mathrm{b}^1(p_r). \tag{17a}$$

As marginal one-particle momentum PDFs, we will now consider the η-generalized Jüttner-Maxwell distributions [22,65], reading

$$f_\mathrm{b}^1(p) = \frac{\mathcal{N}_\eta}{\epsilon(p)^\eta} \exp\left(-\frac{\epsilon(p)}{kT}\right), \quad \eta \geq 0, \tag{17b}$$

where $p \in (-\infty, +\infty)$, and $\epsilon(p)$ denotes the relativistic kinetic energy of a heat bath particle. The normalization constant $\mathcal{N}_\eta$ is determined by the condition

$$1 = \int_{-\infty}^{\infty} dp\, f_\mathrm{b}^1(p). \tag{17c}$$

For $\eta = 0$, Eq. (17b) reduces to the standard Jüttner-Maxwell distribution [65]. On the other hand, as discussed recently [22,38], the PDF with $\eta = 1$ appears to be conserved in relativistic elastic binary collisions. In general, however, the arguments and results presented below remain valid for arbitrary one-particle momentum PDFs $f_\mathrm{b}^1(p)$, i.e., also for momentum distributions other than the η-generalized Jüttner PDFs (17b).

2. Relativistic collision kinematics

Using natural units such that $c = 1$, the relativistic kinetic energy, momentum, and velocity are related by

$$p = mv\gamma(v), \quad \epsilon(p) = (m^2 + p^2)^{1/2}, \tag{18a}$$

$$P = MV\gamma(V), \quad E(P) = (M^2 + P^2)^{1/2}, \tag{18b}$$

where $\gamma(v) \equiv (1 - v^2)^{-1/2}$. As before, capital letters refer to the Brownian particle. Inserting Eqs. (18) into the conservation laws (2), and solving for $\hat{P}$, one finds [22]

$$\hat{P} = \frac{2uE - (1 + u^2)P}{1 - u^2}, \tag{19}$$

where

$$u(p, P) = \frac{P + p}{E + \epsilon} \tag{20}$$

is the center-of-mass velocity. Hence, the momentum change $\Delta P = \hat{P} - P$ of the Brownian particle in a single collision is given by

$$\Delta P = -\frac{2}{1 - u^2} \frac{\epsilon}{E + \epsilon} P + \frac{2}{1 - u^2} \frac{E}{E + \epsilon} p. \tag{21}$$

In the nonrelativistic limit case, where $u^2 \ll 1$, $E \simeq M$, and $\epsilon \simeq m$, this reduces to Eq. (4).

B. Derivation of the Langevin equation

Inserting Eq. (21) into Eq. (5), one obtains the relativistic analog of Eq. (9) as

$$\begin{aligned} \frac{\delta P(t)}{\tau} \simeq &- \left[\frac{1}{\tau}\sum_{r=1}^N \frac{2}{1 - u_r^2} \frac{\epsilon_r}{E + \epsilon_r} I_r(t, \tau)\right] P \\ &+ \frac{1}{\tau}\sum_{r=1}^N \frac{2}{1 - u_r^2} \frac{E}{E + \epsilon_r} p_r I_r(t, \tau), \end{aligned} \tag{22}$$

where $u_r \equiv u(p_r, P)$ and $\epsilon_r \equiv \epsilon(p_r)$. Formally, the collision indicator $I_r(t, \tau)$ is still determined by Eqs. (6) and (7), but differences arise due to the fact that we have to use $V = P/(M^2 + P^2)^{1/2}$ and a relativistic bath distribution now.

Analogous to the nonrelativistic case, we can identify the first term on the RHS of Eq. (22) as friction, and introduce an averaged friction coefficient by

$$\begin{aligned} \nu(P) &\equiv \frac{1}{\tau}\sum_{r=1}^N \left\langle \frac{2}{1 - u_r^2} \frac{\epsilon_r}{E + \epsilon_r} I_r(t, \tau) \right\rangle_\mathrm{b} \\ &= \frac{N}{\tau} \left\langle \frac{2}{1 - u_r^2} \frac{\epsilon_r}{E + \epsilon_r} I_r(t, \tau) \right\rangle_\mathrm{b}, \end{aligned} \tag{23}$$

for some $r \in \{1, \ldots, N\}$. Next, applying a product approximation similar to (8), we obtain

$$\begin{aligned} \nu(P) &\simeq \frac{N}{\tau} \left\langle \frac{2}{1 - u_r^2} \frac{\epsilon_r}{E + \epsilon_r} \right\rangle_\mathrm{b} \langle I_r(t, \tau) \rangle_\mathrm{b} \\ &\overset{(7)}{=} n_\mathrm{b} C(P) \left\langle \frac{2}{1 - u_r^2} \frac{\epsilon_r}{E + \epsilon_r} \right\rangle_\mathrm{b}, \end{aligned} \tag{24}$$

where $n_\mathrm{b} = N/L$ is the density of the heat bath particles, and $C(P)$ is determined by Eq. (7b). Figure 1 shows the P de-

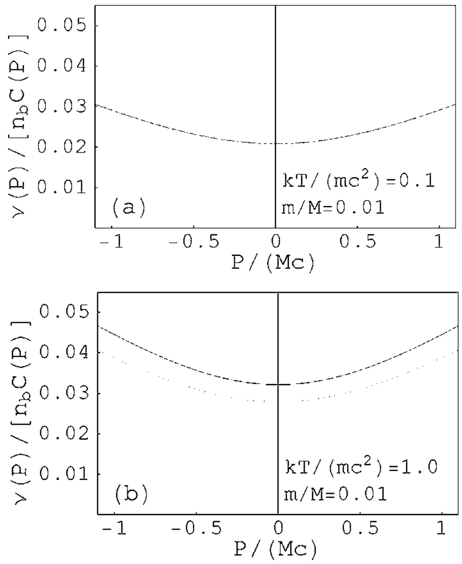

FIG. 1. The momentum-dependent, relativistic friction coefficient $\nu(P)$, divided by the total mean collision rate, $\nu(P)/[n_{\rm b}C(P)]$, as calculated numerically for two different heat bath distributions $f_{\rm b}^1(p)$ and two different bath temperatures, is depicted versus the scaled momentum P. The solid lines refer to the standard Jüttner distribution with $\eta=0$, and the dotted lines to $\eta=1$ in Eq. (17b). (a) *Weakly relativistic heat bath.* In the limit $kT \ll mc^2$ the bath distributions (17b) approach a Maxwellian, and therefore the results for different η practically coincide. In particular, for $P=0$ the nonrelativistic result is recovered. (b) *Strongly relativistic heat bath.* The friction coefficient increases with the temperature of the heat bath.

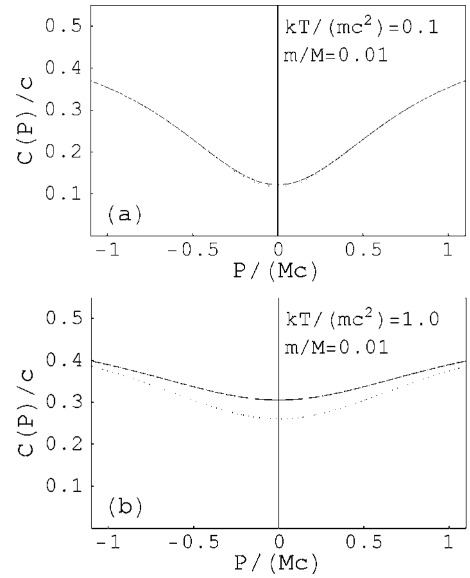

FIG. 2. Relativistic one-particle collision coefficient $C(P) = \langle I_r(t,\tau)\rangle_{\rm b} L/\tau$, numerically calculated for the same parameters as in Fig. 1. The solid lines refer to a standard Jüttner distribution with $\eta=0$, and the dotted lines to $\eta=1$ in Eq. (17b). (a) *Weakly relativistic heat bath.* At small temperatures, the zero-value $C(0)$ is approximately equal to the nonrelativistic Stokes value $\sqrt{kT/(2\pi m)}$. (b) *Strongly relativistic heat bath.* For $|P|\to\infty$ the coefficient $C(P)$ converges to $1/2$.

pendence of $\nu(P)/[n_{\rm b}C(P)]$ for the bath distributions from Eq. (17b). This momentum dependence is induced by the appearance of $u_r=u(p_r,P)$ and $E=E(P)$ in the expectation value on the RHS of Eq. (24). Furthermore, the shape of the one-particle collision coefficient $C(P)=\langle I_r(t,\tau)\rangle_{\rm b}L/\tau$ is depicted in Fig. 2. As one would intuitively expect, the friction coefficient grows with the temperature T of the heat bath (at constant P) as well as with the absolute momentum of the Brownian particle (at constant T). In the nonrelativistic limit case, where $u^2 \ll 1$, $E \simeq M$, and $\epsilon \simeq m$, the relativistic friction coefficient $\nu(P)$ from Eq. (24) reduces to the nonrelativistic result $\nu_0(P)$ from Eq. (11).

At this point, it might be worthwhile to emphasize once again that product approximations of the form

$$\langle G(x_r,p_r) I_r(t,\tau)\rangle_{\rm b} \simeq \langle G(x_r,p_r)\rangle_{\rm b} \langle I_r(t,\tau)\rangle_{\rm b}, \tag{25}$$

as employed in Eq. (8) and also in the first line of Eq. (24), can in principle be omitted by using the explicit representation (6) of the collision indicator and Eq. (A6) of the Appendix; if one opts to avoid such approximations then the accuracy of the Langevin model increases (note that this statement applies to the nonrelativistic case, too). However, in the following we shall continue to use Eq. (25) in order to obtain a relativistic LE that is on an equal footing with the nonrelativistic LE (15).

For this purpose, we interpret the second term on the RHS of Eq. (22) as "noise," defining

$$\chi(P,t) \equiv \frac{1}{\tau}\sum_{r=1}^{N} \frac{2}{1-u_r^2}\frac{E}{E+\epsilon_r} p_r I_r(t,\tau). \tag{26}$$

Averaging over the bath distribution $f_{\rm b}^N$, one finds for the mean value

$$\begin{aligned}\mu(P) \equiv \langle \chi(P,t)\rangle_{\rm b} &= \frac{N}{\tau}\left\langle \frac{2}{1-u_r^2}\frac{E}{E+\epsilon_r} p_r I_r(t,\tau)\right\rangle_{\rm b} \\ &\overset{(25)}{\simeq} \frac{N}{\tau}\left\langle \frac{2}{1-u_r^2}\frac{E}{E+\epsilon_r} p_r\right\rangle_{\rm b} \langle I_r(t,\tau)\rangle_{\rm b} \\ &\simeq n_{\rm b}C(P)\left\langle \frac{2}{1-u_r^2}\frac{E}{E+\epsilon_r} p_r\right\rangle_{\rm b}.\end{aligned} \tag{27}$$

In contrast to the nonrelativistic case, the mean value μ of the relativistic Langevin force $\chi(t)\equiv\chi(P,t)$ depends on the momentum P of the Brownian particle. This can be attributed to the appearance of $u_r^2=(P+p_r)^2/(E+\epsilon_r)^2$ in Eq. (27). As shown in Fig. 3, the quantity $\mu(P)/[n_{\rm b}C(P)]$ is positive for $P>0$ and negative for $P<0$. Thus, on average, the relativistic stochastic force tends to accelerate particles in the direction of their motion, but this effect is compensated by the increase of the friction coefficient $\nu(P)$ at high values of P (cf. Fig. 1).

Let us next take a closer look at the covariance function

$$\sigma_{ts} \equiv \langle [\chi(t)-\langle\chi(t)\rangle_{\rm b}][\chi(s)-\langle\chi(s)\rangle_{\rm b}]\rangle_{\rm b}. \tag{28}$$

In the nonrelativistic case, the stochastic force possesses a vanishing mean value $\langle\xi(t)\rangle_{\rm b}=0$. According to Eq. (27), this is no longer the case for the relativistic noise

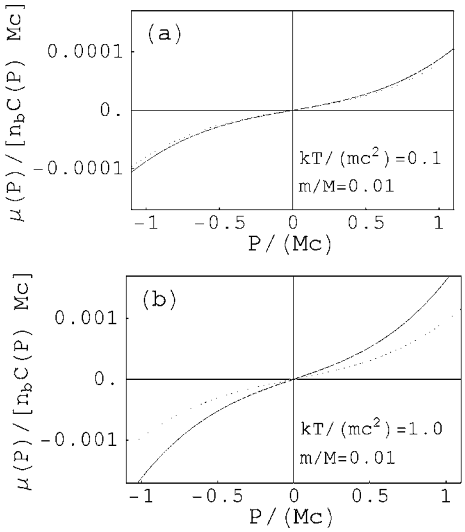

FIG. 3. Mean value of the relativistic stochastic force $\mu(P)\equiv\langle\chi(t)\rangle_{\mathrm{b}}$ calculated numerically for two different heat bath distributions $f_{\mathrm{b}}^{1}(p)$ and two different bath temperatures. Solid lines refer to a standard Jüttner distribution with $\eta=0$, and dotted lines to $\eta=1$ in Eq. (17b).

force $\langle\chi(t)\rangle_{\mathrm{b}}\neq 0$. In order to explicitly calculate σ_{ts} for the relativistic case, it is convenient to introduce the abbreviation

$$\kappa_r = \frac{2}{1-u_r^2}\frac{E}{E+\epsilon_r}p_r.$$

Assuming, as before, that collisions can be viewed as independent events, the correlation function (28) vanishes at nonequal times $t\neq s$, and we thus find

$$\begin{aligned}
\sigma_{ts} &\overset{(27)}{\simeq} \delta_{ts}\left\langle\left(\frac{1}{\tau}\sum_{r=1}^{N}\kappa_r I_r(t,\tau)\right)^2-\mu^2(P)\right\rangle_{\mathrm{b}}\\
&\overset{(25)}{\simeq} \frac{\delta_{ts}}{\tau^2}\left(\sum_{r=1}^{N}\langle\kappa_r^2\rangle_{\mathrm{b}}\langle I_r(t,\tau)\rangle_{\mathrm{b}}-\tau^2\mu^2(P)\right.\\
&\quad\left.\times\sum_{r=1}^{N}\sum_{j\neq r}^{N}\langle\kappa_r\rangle_{\mathrm{b}}\langle I_r(t,\tau)\rangle_{\mathrm{b}}\langle\kappa_j\rangle_{\mathrm{b}}\langle I_j(t,\tau)\rangle_{\mathrm{b}}\right)\\
&= \frac{\delta_{ts}}{\tau^2}\sum_{r=1}^{N}\left(\langle\kappa_r^2\rangle_{\mathrm{b}}\langle I_r(t,\tau)\rangle_{\mathrm{b}}-\frac{\tau^2}{N}\mu^2(P)\right.\\
&\quad\left.+\langle I_r(t,\tau)\rangle_{\mathrm{b}}\langle\kappa_r\rangle_{\mathrm{b}}\sum_{j\neq r}^{N}\langle\kappa_j\rangle_{\mathrm{b}}\langle I_j(t,\tau)\rangle_{\mathrm{b}}\right)\\
&\overset{(27)}{=} \frac{\delta_{ts}}{\tau^2}\sum_{r=1}^{N}\left(\langle\kappa_r^2\rangle_{\mathrm{b}}\langle I_r(t,\tau)\rangle_{\mathrm{b}}-\frac{\tau^2}{N}\mu^2(P)+\frac{\tau}{N}\mu(P)\sum_{j\neq r}^{N}\frac{\tau}{N}\mu(P)\right).
\end{aligned}\tag{29}$$

From this, we obtain

$$\begin{aligned}
\sigma_{ts} &\simeq \frac{\delta_{ts}}{\tau}\frac{N}{\tau}\langle\kappa_r^2\rangle_{\mathrm{b}}\langle I_r(t,\tau)\rangle_{\mathrm{b}}-\frac{\delta_{ts}}{N}\mu^2(P)\\
&\overset{(7a)}{=} \frac{\delta_{ts}}{\tau}n_{\mathrm{b}}C(P)\langle\kappa_r^2\rangle_{\mathrm{b}}-\frac{\delta_{ts}}{N}\mu^2(P).
\end{aligned}\tag{30}$$

The last term vanishes, if we consider the thermodynamic limit (TDL) of an infinite heat bath, i.e., $N,L\to\infty$ such that $n_{\mathrm{b}}=N/L=$const. Thus, reinserting the explicit expression for κ_r, we obtain in this limit

$$\sigma_{ts}\to\frac{\delta_{ts}}{\tau}n_{\mathrm{b}}C(P)\left\langle\left(\frac{2}{1-u_r^2}\frac{E}{E+\epsilon_r}p_r\right)^2\right\rangle_{\mathrm{b}}. \tag{31}$$

In principle, any higher correlation function can be calculated in the same manner. It is also evident that the noise force is non-Gaussian, because the relativistic bath distribution $f_{\mathrm{b}}(p_r)$ that determines the averages $\langle\cdot\rangle_{\mathrm{b}}$—and, thus, the noise correlations—is non-Gaussian.

Finally, by substituting the averaged friction coefficient

$$\nu(P)=n_{\mathrm{b}}C(P)\left\langle\frac{2}{1-u_r^2}\frac{\epsilon_r}{E+\epsilon_r}\right\rangle_{\mathrm{b}} \tag{32a}$$

for the term in square brackets in Eq. (22), imposing the TDL for the bath, and letting $\tau\to 0$ in Eq. (22), we obtain the relativistic LE

$$\dot{P}=-\nu(P)P+\chi(t), \tag{32b}$$

where, in view of the approximation (25), the non-Gaussian momentum-dependent noise force $\chi(t)\equiv\chi(P,t)$ is characterized by the mean

$$\mu(P)\equiv\langle\chi(t)\rangle_{\mathrm{b}}=n_{\mathrm{b}}C(P)\left\langle\frac{2}{1-u_r^2}\frac{E}{E+\epsilon_r}p_r\right\rangle_{\mathrm{b}} \tag{32c}$$

and the covariance

$$\sigma(t,s)\equiv\langle[\chi(t)-\langle\chi(t)\rangle_{\mathrm{b}}][\chi(s)-\langle\chi(s)\rangle_{\mathrm{b}}]\rangle_{\mathrm{b}}=2D(P)\delta(t-s), \tag{32d}$$

with the (momentum-space) diffusion coefficient given by

$$D(P)=\frac{n_{\mathrm{b}}}{2}C(P)\left\langle\left(\frac{2}{1-u_r^2}\frac{E}{E+\epsilon_r}p_r\right)^2\right\rangle_{\mathrm{b}}. \tag{32e}$$

In Fig. 4 the ratio $D(P)/[n_{\mathrm{b}}C(P)]$ is plotted for the same parameters as in Figs. 1 and 3. As it is evident from the diagrams, this quantity increases with temperature T and absolute momentum P of the Brownian particle.

IV. SUMMARY

We conclude the derivation of the relativistic LE with a set of general remarks.

(i) While deriving the relativistic LE (32), we made use of the stationarity, independence, and homogeneity of the bath distribution (17a); we did not, however, rely on the specific properties of the marginal momentum PDF. Hence, the above results hold true for arbitrary one-particle momentum distributions $f_{\mathrm{b}}^{1}(p)$.

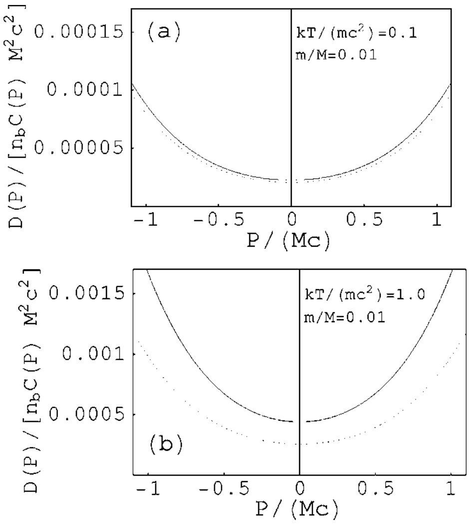

FIG. 4. Relativistic diffusion coefficients $D(P)$ calculated numerically for different heat bath distributions $f_{\mathrm{b}}^{1}(p)$ for (a) weakly and (b) strongly relativistic heat. The solid lines refer to a standard Jüttner distribution with $\eta=0$, and the dotted lines to $\eta=1$ in Eq. (17b).

(ii) In order to be able to use the LEs (32) derived above, one still needs to calculate the mean collision rate $C(P)/L$, which is determined by Eq. (7b); see the Appendix. We also emphasize once again that the approximation (25), leading to the appearance of $C(P)$, can in principle be omitted (in the nonrelativistic as well as in the relativistic case). More precise results for friction coefficients and noise correlations can then be extracted from Eq. (A6) in the Appendix.

(iii) The stochastic force $\chi(t)$ in Eqs. (32) is δ correlated (memory-free), but non-Gaussian; i.e., in order to completely specify the stochastic process one actually has to determine all higher order correlation functions. This is practically unfeasible. Therefore, in numerical studies and/or practical applications, one could use a Gaussian approximation (GA) of Eqs. (32), obtained in the following manner. We rewrite Eq. (32b) equivalently as

$$\dot{P} = -\bar{\nu}(P)P + \sqrt{2D(P)}\bar{\zeta}(t), \tag{33a}$$

where

$$\bar{\nu}(P) \equiv \nu(P) - \frac{\mu(P)}{P}, \quad \bar{\zeta}(t) \equiv \frac{\chi(t) - \mu(P)}{\sqrt{2D(P)}}. \tag{33b}$$

Reminiscent of standardized Gaussian white noise, the effective noise force $\bar{\zeta}(t)$ is characterized by

$$\langle \bar{\zeta}(t) \rangle = 0, \quad \langle \bar{\zeta}(t)\bar{\zeta}(s) \rangle = \delta(t-s), \tag{33c}$$

but the higher moments are non-Gaussian. Accordingly, the GA is achieved by replacing $\bar{\zeta}(t)$ in Eq. (33a) with standardized momentum-independent Gaussian white noise $\zeta(t)$. The resulting stochastic differential equation is a standard LE with multiplicative Gaussian white noise. Hence, after having specified a discretization rule, one can easily write down the corresponding Fokker-Planck equation as well as the corresponding stationary distribution [64].

The GA obtained in this way neglects higher-order cumulants of the noise, so it cannot be expected that the "truncated" LE yields exactly the same relaxation behavior and/or the same stationary solution as the full relativistic LE (32) [63,66]. Nevertheless, this approximation should provide useful estimates. In particular, if the stationary momentum distribution of the Brownian particle can be guessed by other arguments [22], then the GA can be made self-consistent with respect to this distribution by fixing a suitably generalized Einstein relation for the friction and noise coefficients. In this case it suffices to calculate, e.g., $\bar{\nu}(P)$, because the corresponding noise amplitude $\bar{D}(P)$ is then uniquely determined by the Einstein relation (i.e., by the stationary distribution).

Due to the multiplicative noise coupling, the results obtained from the GA will also depend on the choice of the discretization rule [54,57–64]. Loosely speaking, this discretization dilemma is the price that one has to pay for mapping the large number of collisions between t and $t+\tau$ onto a single instant of time. Our experience with the nonrelativistic LE (cf. remarks at the end of Sec. II B) suggests that the "transport" or "kinetic" interpretation, corresponding to the post-point-discretization rule [54,62–64], should be preferable in the relativistic case as well.

(iv) In principle, it should be straightforward to generalize the above approach to higher space dimensions, by expressing the momentum vector after the collision, $\hat{\boldsymbol{P}}$, in terms of the momenta before the collision, $\boldsymbol{P}$ and $\boldsymbol{p}$, analogous to Eq. (21). In the 2D or 3D case, complications may arise mostly due to the fact that one also has take into account the corresponding collision angles and cross sections (e.g., when determining the collision rates; see the comments at the end of the Appendix).

(v) According to our above results, the previously proposed "relativistic" LEs [21,26,27,30] should be viewed as approximations, which can be fruitful for generating or simulating ensembles of relativistic particles in a simple manner. It is also evident now why these earlier approaches were intrinsically limited. Debbasch *et al.* [30] have postulated that the relativistic stochastic force in the rest frame of the bath is ordinary Gaussian white noise with a constant amplitude D, whereas we in our prior work [21,27] assumed Gaussian noise in the (comoving) rest frame of the Brownian particle. As follows from the derivation presented here, neither of these assumptions is accurate if one properly takes into account both the relativistic conservation laws and the relativistic momentum distributions of the heat bath particles. However, if we consider suitably chosen momentum-dependent friction and diffusion coefficients, then the previously proposed relativistic LEs [21,26,27,30] become equivalent to the Gaussian approximation of the relativistic LE (32).

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APPENDIX: CALCULATION OF THE COLLISION RATE

We aim to derive an explicit expression for the expectation value $\langle I_r(t,\tau)\rangle_{\mathrm{b}}$ in the limit $\tau\to 0$, as, e.g., required in Eqs. (10).

By definition, the function $I_r(t,\tau)\in\{0,1\}$ indicates whether or not the Brownian particle has collided with the heat bath particle r during the time interval $[t,t+\tau]$. The positions of the Brownian and heat bath particles at time t are denoted by X and x_r, respectively. Ignoring the possibility of a collision, for small enough τ, the new positions at time $t+\tau$ would be given by

$$X' = X + V\tau, \quad x_r' = x_r + v_r\tau, \tag{A1}$$

where V and v_r are the velocities. Then, the indicator function $I_r(t,\tau)$ can be explicitly represented as

$$\begin{aligned} I_r(t,\tau) &= \Theta(X-x_r)\Theta(x_r'-X')\Theta(v_r-V) \\ &\quad + \Theta(x_r-X)\Theta(X'-x_r')\Theta(V-v_r), \end{aligned} \tag{A2}$$

where $\Theta(x)$ is the Heaviside function, defined by

$$\Theta(x) = \begin{cases} 0, & x<0, \\ 1/2, & x=0, \\ 1, & x>0. \end{cases} \tag{A3}$$

The first (second) summand in Eq. (A2) refers to the initial configuration, where the heat bath particle is located at the left (right) side of the Brownian particle. Let us list some properties of the collision indicator $I_r(t,\tau)$.

First we note that $I_r(t,\tau)$ is idempotent, i.e.,

$$I_r^j(t,\tau) = I_r(t,\tau) \tag{A4a}$$

holds for $j=1,2\ldots$. Furthermore, for $\tau\to 0$, we have

$$I_r(t,0) = 0. \tag{A4b}$$

Accordingly, the Taylor expansion at $\tau=0$ gives

$$I_r(t,\tau) \simeq \left[\frac{\partial I_r}{\partial\tau}(t,0)\right]\tau. \tag{A4c}$$

In order to determine $\left\langle \frac{\partial I_r}{\partial\tau}(t,0)\right\rangle_{\mathrm{b}}$, we note that

$$\begin{aligned} \frac{\partial}{\partial\tau}\Theta(x_r'-X')\bigg|_{\tau=0} &= \frac{\partial}{\partial\tau}\Theta(x_r - X + (v_r-V)\tau)\bigg|_{\tau=0} \\ &= (v_r-V)\,\delta(x_r-X+(v_r-V)\tau)|_{\tau=0} \\ &= (v_r-V)\,\delta(x_r-X), \end{aligned}$$

and, analogously,

$$\begin{aligned} \frac{\partial}{\partial\tau}\Theta(X'-x_r')\bigg|_{\tau=0} &= \frac{\partial}{\partial\tau}\Theta(X - x_r + (V-v_r)\tau)\bigg|_{\tau=0} \\ &= (V-v_r)\,\delta(X-x_r+(V-v_r)\tau)|_{\tau=0} \\ &= (V-v_r)\,\delta(X-x_r). \end{aligned}$$

Hence, we find

$$\begin{aligned} \frac{\partial I_r}{\partial\tau}(t,0) &= (v_r-V)\Theta(X-x_r)\,\delta(x_r-X)\Theta(v_r-V) \\ &\quad + (V-v_r)\Theta(x_r-X)\,\delta(X-x_r)\Theta(V-v_r), \\ &= \Theta(0)(v_r-V)\,\delta(x_r-X)\Theta(v_r-V) \\ &\quad + \Theta(0)(V-v_r)\,\delta(X-x_r)\Theta(V-v_r), \end{aligned}$$

and, with $\Theta(0)=1/2$, the useful result

$$\begin{aligned} \frac{\partial I_r}{\partial\tau}(t,0) &= \frac{1}{2}(v_r-V)\,\delta(x_r-X)\Theta(v_r-V) \\ &\quad + \frac{1}{2}(V-v_r)\,\delta(X-x_r)\Theta(V-v_r). \end{aligned} \tag{A4d}$$

Now let us consider a spatially homogeneous one-particle bath distribution of the form

$$\tilde{f}_{\mathrm{b}}^1(x_r,v_r) = \frac{1}{L}\tilde{f}_{\mathrm{b}}^1(v_r)\begin{cases} 1, & x_r\in[-L/2,L/2], \\ 0, & x_r\notin[-L/2,L/2], \end{cases} \tag{A5}$$

and some function $\tilde{G}(x_r,v_r)$ such that the expectation value $\langle\tilde{G}(x_r,v_r)\rangle_{\mathrm{b}}$ exists. We are interested in expectations of the form

$$\langle\tilde{G}(x_r,v_r)I_r^j(t,\tau)\rangle_{\mathrm{b}} \overset{(A4a)}{=} \langle\tilde{G}(x_r,v_r)I_r(t,\tau)\rangle_{\mathrm{b}},$$

as required for calculating the mean value of the stochastic force and its higher correlation functions [e.g., compare first line of Eq. (29)]. For small τ, we may truncate the Taylor expansion after the linear term, yielding

$$\langle\tilde{G}(x_r,v_r)I_r(t,\tau)\rangle_{\mathrm{b}} \simeq \left\langle \tilde{G}(x_r,v_r)\frac{\partial I_r}{\partial\tau}(t,0)\right\rangle_{\mathrm{b}}\tau. \tag{A6a}$$

Making use of the result (A4d), the mean value on the RHS is given by

$$\begin{aligned} &\left\langle \tilde{G}(x_r,v_r)\frac{\partial I_r}{\partial\tau}(t,0)\right\rangle_{\mathrm{b}} \\ &= \frac{1}{2L}\int_V^\infty dv_r(v_r-V)\times\tilde{G}(X,v_r)\tilde{f}_{\mathrm{b}}^1(v_r) \\ &\quad + \frac{1}{2L}\int_{-\infty}^V dv_r(V-v_r)\times\tilde{G}(X,v_r)\tilde{f}_{\mathrm{b}}^1(v_r). \end{aligned} \tag{A6b}$$

In particular, by choosing $\tilde{G}(x_r,v_r)\equiv 1$, we find the collision rate

$$\lim_{\tau\to 0}\frac{\langle I_r(t,\tau)\rangle_{\mathrm{b}}}{\tau} = \left\langle\frac{\partial I_r}{\partial\tau}(t,0)\right\rangle_{\mathrm{b}} = \frac{1}{L}\tilde{C}(V), \tag{A7a}$$

where

$$\tilde{C}(V) \equiv \frac{1}{2}\int_V^\infty dv_r(v_r-V)\tilde{f}_{\mathrm{b}}^1(v_r) + \frac{1}{2}\int_{-\infty}^V dv_r(V-v_r)\tilde{f}_{\mathrm{b}}^1(v_r). \tag{A7b}$$

The following comments are in order.

(i) The above derivation is valid for both nonrelativistic and relativistic heat bath distributions $\tilde{f}_{\rm b}^{1}(v_r)$. Upon identifying $C(P) \equiv \tilde{C}(V(P))$, where P is the momentum of the Brownian particle, we obtain the rigorous justification for Eq. (7a). However, in the nonrelativistic case we have $V = P/M$, whereas in the relativistic case $V = P/(M^2+P^2)^{1/2}$. Additionally, we note that the support interval of the relativistic velocity distribution $\tilde{f}_{\rm b}^{1}(v_r)$ is given by $[-c,c]$, which determines the effective upper and lower integral boundaries in Eq. (A7b)

(ii) Given a certain bath distribution $\tilde{f}_{\rm b}^{1}(v_r)$, the exact result (A6) allows for evaluating the quality of the product approximations (8) and (25), respectively.

(iii) The Stokes approximation corresponds to setting $V=0$ in Eq. (A7b), yielding

$$\tilde{C}(0) = \frac{1}{2}\int_{-\infty}^{\infty} dv_r |v_r| \tilde{f}_{\rm b}^{1}(v_r). \tag{A8}$$

This shows that the Stokes approximation is useful for slow Brownian particles, but inappropriate at high velocities.

(iv) It is in principle possible to apply the same procedure to higher space dimensions, but then the expression (A2) for the indicator unction has to be modified accordingly (e.g., by taking into account the geometric shape of the Brownian particle). As a consequence, analytic calculations will become much more difficult.

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